Algebras generated by reciprocals of linear forms

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Abstract

Let Δ be a finite set of nonzero linear forms in several variables with coefficients in a field $\mathbf{K}$ of characteristic zero. Consider the $\mathbf{K}$-algebra $C(\Delta)$ of rational functions generated by $\{1/\alpha \mid \alpha \in \Delta\}$. Then the ring $\partial(V)$ of differential operators with constant coefficients naturally acts on $C(\Delta)$. We study the graded $\partial(V)$-module structure of $C(\Delta)$. We especially find standard systems of minimal generators and a combinatorial formula for the Poincaré series of $C(\Delta)$. Our proofs are based on a theorem by Brion-Vergne [BrV] and results by Orlik-Terao [OrT2].

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1 Introduction and main results

Let V be a vector space of dimension ℓ over a field $\mathbf{K}$ of characteristic zero. Let Δ be a finite subset of the dual space V^* of V. We assume that Δ does not contain the zero vector and that no two vectors are proportional throughout this paper. Let $S = S(V^*)$ be the symmetric algebra of V^*. It is regarded as the algebra of polynomial functions on V. Let $S_{(0)}$ be the field of quotients of S, which is the field of rational functions on V.

Definition 1.1. *Let $C(\Delta)$ be the $\mathbf{K}$-subalgebra of $S_{(0)}$ generated by the set*

$$\{\frac{1}{\alpha} \mid \alpha \in \Delta\}.$$

Regard $C(\Delta)$ as a graded $\mathbf{K}$-algebra with $\deg(1/\alpha) = 1$ for $\alpha \in \Delta$.

Definition 1.2. *Let $\partial(V)$ be the $\mathbf{K}$-algebra of differential operators with constant coefficients. Agree that the constant multiplications are in $\partial(V)$: $\mathbf{K} \subset \partial(V)$.*

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If $x_1, \cdots, x_\ell$ are a basis for V^*, then $\partial(V)$ is isomorphic to the polynomial algebra $\mathbf{K}[\partial/\partial x_1, \cdots, \partial/\partial x_\ell]$. Regard $\partial(V)$ as a graded $\mathbf{K}$-algebra with $\deg(\partial/\partial x_i) = 1 \quad (1 \le i \le \ell)$. It naturally acts on $S_{(0)}$. We regard $C(\Delta)$ as a graded $\partial(V)$-module. In this paper we study the $\partial(V)$-module structure of $C(\Delta)$. In particular, we find systems of minimal generators (Theorem 1.3) and a combinatorial formula for the Poincaré (or Hilbert) series $\mathrm{Poin}(C(\Delta), t)$ of $C(\Delta)$ (Theorem 1.4).

In order to present our results we need several definitions. Let $\mathbf{E}_p(\Delta)$ be the set of all p-tuples composed of elements of Δ. Let $\mathbf{E}(\Delta) := \bigcup_{p\ge 0} \mathbf{E}_p(\Delta)$. The union is disjoint. Write $\prod \mathcal{E} := \alpha_1 \ldots \alpha_p \in S$ when $\mathcal{E} = (\alpha_1, \ldots, \alpha_p) \in \mathbf{E}_p(\Delta)$. Then one can write

$$C(\Delta) = \sum_{\mathcal{E}\in\mathbf{E}(\Delta)} \mathbf{K}\left(\prod \mathcal{E}\right)^{-1}.$$

Let

$$\begin{aligned} \mathbf{E}^i(\Delta) &= \{\mathcal{E} \in \mathbf{E}(\Delta) \mid \mathcal{E} \text{ is linearly independent}\}, \\ \mathbf{E}^d(\Delta) &= \{\mathcal{E} \in \mathbf{E}(\Delta) \mid \mathcal{E} \text{ is linearly dependent}\}. \end{aligned}$$

Note that $\mathcal{E} \in \mathbf{E}^d(\Delta)$ if $\mathcal{E}$ contains a repetition. In a special lecture at the Japan Mathematical Society in 1992, K. Aomoto suggested the study of the finite-dimensional graded $\mathbf{K}$-vector space

$$AO(\Delta) := \sum_{\mathcal{E}\in\mathbf{E}^i(\Delta)} \mathbf{K}\left(\prod \mathcal{E}\right)^{-1}.$$

Let

$$\mathcal{A}(\Delta) = \{\ker(\alpha) \mid \alpha \in \Delta\}.$$

Then $\mathcal{A}(\Delta)$ is a (central) arrangement of hyperplanes [OrT1] in V. K. Aomoto conjectured, when $\mathbf{K} = \mathbf{R}$, that the dimension of $AO(\Delta)$ is equal to the number of connected components of

$$M(\mathcal{A}(\Delta)) := V \setminus \bigcup_{H\in\mathcal{A}(\Delta)} H.$$

This conjecture was verified in [OrT2], where explicit $\mathbf{K}$-bases for $AO(\Delta)$ were constructed. This paper can be considered as a sequel to [OrT2]. (It should be remarked that constructions in [OrT2] were generalized for oriented matroids by R. Cordovil [Cor].) We will prove the following

Theorem 1.3. *Let $\mathcal{B}$ be a $\mathbf{K}$-basis for $AO(\Delta)$. Let $\partial(V)_+$ denote the maximal ideal of $\partial(V)$ generated by the homogeneous elements of degree one. Then*

(1) the set $\mathcal{B}$ is a system of minimal genarators for the $\partial(V)$-module $C(\Delta)$,

(2) $C(\Delta) = \partial(V)_+ C(\Delta) \oplus AO(\Delta)$, and

(3) $\partial(V)_+ C(\Delta) = \sum_{\mathcal{E}\in\mathbf{E}^d(\Delta)} \mathbf{K}\left(\prod \mathcal{E}\right)^{-1}$. In paticular, $\partial(V)_+ C(\Delta)$ is an ideal of $C(\Delta)$.

Let $\mathrm{Poin}(\mathcal{A}(\Delta), t)$ be the Poincaré polynomial [OrT1, Definition 2.48] of $\mathcal{A}(\Delta)$. (It is defined combinatorially and is known to be equal to the the Poincaré polynomial of $M(\mathcal{A}(\Delta))$ when $\mathbf{K} = \mathbf{C}$ [OrS] [OrT1, Theorem 5.93].) Then we have

Theorem 1.4. *The Poincaré series* $\mathrm{Poin}(C(\Delta), t)$ *of the graded module* $C(\Delta)$ *is equal to* $\mathrm{Poin}(\mathcal{A}(\Delta), (1-t)^{-1}t)$.

In order to prove these theorems we essentially use a theorem by M. Brion and M. Vergne [BrV, Theorem1] and results from [OrT2]. By Theorem 1.4 and the factorization theorem (Theorem 2.4) in [Ter1], we may easily show the following two corollaries:

Corollary 1.5. *If* $\mathcal{A}(\Delta)$ *is a free arrangement with exponents* $(d_1, \cdots, d_\ell)$ *[Ter1] [OrT1, Definitions 4.15, 4.25], then*

$$\mathrm{Poin}(C(\Delta), t) = (1-t)^{-\ell} \prod_{i=1}^{\ell} \{1 + (d_i - 1)t\}.$$

Example 1.6. *Let* $x_1, \ldots, x_\ell$ *be a basis for* V^*. *Let* $\Delta = \{x_i - x_j \mid 1 \le i < j \le \ell\}$. *Then* $\mathcal{A}(\Delta)$ *is known to be a free arrangement with exponents* $(0, 1, \ldots, \ell-1)$ *[OrT1, Example 4.32]. So, by Corollary 1.5, we have*

$$\mathrm{Poin}(C(\Delta), t) = (1-t)^{-\ell+1}(1+t)(1+2t)\ldots(1+(\ell-2)t).$$

For example, when ℓ*=3, we have*

$$\begin{aligned} & \mathrm{Poin}\left(\mathbf{K}\left[\frac{1}{x_1 - x_2}, \frac{1}{x_2 - x_3}, \frac{1}{x_1 - x_3}\right], t\right) = (1+t)/(1-t)^2 \\ = \quad & 1 + 3t + 5t^2 + 7t^3 + 9t^4 + \ldots, \end{aligned}$$

which can be easily checked by direct computation.

When $\mathcal{A}(\Delta)$ *is the set of reflecting hyperplanes of any (real or complex) reflection group, Corollary 1.5 can be applied because* $\mathcal{A}(\Delta)$ *is known to be a free arrangement [Sai1] [Ter2].*

Corollary 1.7. *If* $\mathcal{A}(\Delta)$ *is generic (i, e.,* $|\Delta| \ge \ell$ *and any* ℓ *vectors in* Δ *are linearly independent), then*

$$\mathrm{Poin}(C(\Delta), t) = (1-t)^{-\ell} \sum_{i=0}^{\ell-1} \binom{|\Delta| - \ell + i - 1}{i} t^i.$$

2 Proofs

In this section we prove Theorems 1.3 and 1.4. For $\varepsilon \in \mathbf{E}(\Delta)$, let $V(\varepsilon)$ denote the set of common zeros of ε : $V(\varepsilon) = \bigcap_{i=1}^{p} \ker(\alpha_i)$ when $\varepsilon = (\alpha_1, \cdots, \alpha_p)$. Define

$$L = L(\Delta) = \{V(\varepsilon) \mid \varepsilon \in \mathbf{E}(\Delta)\}.$$

Agree that $V(\varepsilon) = V$ if ε is the empty tuple. Introduce a partial order $\leq$ into L by reverse inclusion: $X \leq Y \Leftrightarrow X \supseteq Y$. Then L is equal to the intersection lattice of the arrangement $\mathcal{A}(\Delta)$ [OrT1, Definition 2.1]. For $X \in L$, define

$$\mathbf{E}_X(\Delta) := \{\varepsilon \in \mathbf{E}(\Delta) \mid V(\varepsilon) = X\}.$$

Then

$$\mathbf{E}(\Delta) = \bigcup_{X \in L} \mathbf{E}_X(\Delta) \quad \text{(disjoint)}.$$

Define

$$C_X(\Delta) := \sum_{\varepsilon \in \mathbf{E}_X(\Delta)} \mathbf{K}(\prod \varepsilon)^{-1}.$$

Then $C_X(\Delta)$ is a $\partial(V)$-submodule of $C(\Delta)$. The following theorem is equivalent to Lemma 3.2 in [OrT2]. Our proof is a rephrasing of the proof there.

Proposition 2.1.

$$C(\Delta) = \bigoplus_{X \in L} C_X(\Delta).$$

Proof. It is obvious that $C(\Delta) = \sum_{X \in L} C_X(\Delta)$. Suppose that $\sum_{X \in L} \phi_X = 0$ with $\phi_X \in C_X(\Delta)$. We will show that $\phi_X = 0$ for all $X \in L$. By taking out the degree p part, we may assume that $\deg \phi_X = p$ for all $X \in L$. Let $\mathcal{S} = \{X \in L \mid \phi_X \neq 0\}$. Suppose $\mathcal{S}$ is not empty. Then there exists a minimal element X_0 in $\mathcal{S}$ (with respect to the partial order by reverse inclusion). Let $X \in \mathcal{S} \setminus \{X_0\}$ and write

$$\phi_X = \sum_{\varepsilon \in \mathbf{E}_X(\Delta)} c_\varepsilon (\prod \varepsilon)^{-1}$$

with $c_\varepsilon \in \mathbf{K}$. Let $\varepsilon \in \mathbf{E}_X(\Delta)$. Because of the minimality of X_0, one has $X_0 \not\subseteq X$. Thus there exists $\alpha_0 \in \varepsilon$ such that $X_0 \not\subseteq \ker(\alpha_0)$. Let $I(X_0)$ be the prime ideal of S generated by the polynomial functions vanishing on X_0. Then $\alpha_0 \notin I(X_0)$. Thus

$$(\prod \Delta)^p (\prod \varepsilon)^{-1} \in I(X_0)^{p|\Delta_{X_0}| - p + 1},$$

where $\prod \Delta := \prod_{\alpha \in \Delta} \alpha$ and $\Delta_{X_0} = \Delta \cap I(X_0)$. Multiply $(\prod \Delta)^p$ to the both sides of

$$\phi_{X_0} = - \sum_{\substack{X \in \mathcal{S} \\ X \neq X_0}} \phi_X$$

to get

$$\begin{aligned} (\prod \Delta)^p \phi_{X_0} &= - \sum_{\substack{X \in \mathcal{S} \\ X \neq X_0}} (\prod \Delta)^p \phi_X \\ &= - \sum_{\substack{X \in \mathcal{S} \\ X \neq X_0}} \sum_{\varepsilon \in \mathbf{E}_X(\Delta)} c_\varepsilon (\prod \Delta)^p (\prod \varepsilon)^{-1} \in I(X_0)^{p|\Delta_{X_0}| - p + 1}. \end{aligned}$$

Since $(\prod \Delta)/(\prod \Delta_{X_0}) \in S \setminus I(X_0)$ and $I(X_0)^{p|\Delta_{X_0}|-p+1}$ is a primary ideal, one has

$$(\prod \Delta_{X_0})^p \phi_{X_0} \in I(X_0)^{p|\Delta_{X_0}|-p+1}.$$

This is a contradiction because

$$\deg(\prod \Delta_{X_0})^p \phi_{X_0} = p|\Delta_{X_0}| - p.$$

Therefore $\mathcal{S} = \phi$. □

Next we will study the structure of $C_X(\Delta)$ for each $X \in L$. Let $AO_X(\Delta)$ be the $\mathbf{K}$-subspace of $AO(\Delta)$ generated over $\mathbf{K}$ by

$$\{(\prod \varepsilon)^{-1} \mid \varepsilon \in \mathbf{E}(\Delta)^i \cap \mathbf{E}_X(\Delta)\}.$$

Then

$$AO(\Delta) = \bigoplus_{X \in L} AO_X(\Delta)$$

by Proposition 2.1. Let $\mathcal{B}_X$ be a $\mathbf{K}$-basis for $AO_X(\Delta)$. Then we have

Proposition 2.2. *The $\partial(V)$-module $C_X(\Delta)$ can also be regarded as a free $\partial(V/X)$-module with a basis $\mathcal{B}_X$. In other words, there exists a natural graded isomorphism*

$$\partial(V/X) \otimes_{\mathbf{K}} AO_X(\Delta) \simeq C_X(\Delta).$$

Proof. First assume that Δ spans V^* and $X = \{\mathbf{0}\}$. Then $\mathbf{E}(\Delta)^i \cap \mathbf{E}(\Delta)_X$ is equal to the set of $\mathbf{K}$-bases for V^* which are contained in Δ. Thus $AO_X(\Delta)$ is generated over $\mathbf{K}$ by

$$\{(\prod \varepsilon)^{-1} \mid \varepsilon \in \mathbf{E}_\ell(\Delta) \text{ is a basis for } V\}.$$

Similarly C_X is spanned over $\mathbf{K}$ by

$$\{(\prod \varepsilon)^{-1} \mid \varepsilon \in \mathbf{E}(\Delta) \text{ spans } V\}.$$

Then Theorem 1 of [BrV] is exactly the desired result. Next let $X \in L$ and $\overline{V} = V/X$. Regard the dual vector space $\overline{V}^*$ as a subspace of V^* and the symmetric algebra $\overline{S} := S(\overline{V}^*)$ of $\overline{V}^*$ as a subring of S. Then $\Delta_X := I(X) \cap \Delta$ is a subset of $\overline{V}^*$ and Δ_X spans $\overline{V}^*$. Consider $AO(\Delta_X)$ and $C(\Delta_X)$ which are both contained in $\overline{S}_{(0)}$. Note that $C_X(\Delta)$ can be regarded as a $\partial(V/X)$-module because $\partial(X)$ annihilates $C_X(\Delta)$. Denote the zero vector of $\overline{V}$ by $\overline{X}$. Then it is not difficult to see that

$$C_{\overline{X}}(\Delta_X) \simeq C_X(\Delta) \quad (\text{as } \partial(\overline{V})\text{-modules}),$$

$$AO_{\overline{X}}(\Delta_X) \simeq AO_X(\Delta) \quad (\text{as } \mathbf{K}\text{-vector spaces}).$$

Since there exists a natural graded isomorphism

$$C_{\overline{X}}(\Delta_X) \simeq \partial(\overline{V}) \otimes_{\mathbf{K}} AO_{\overline{X}}(\Delta_X),$$

one has

$$C_X(\Delta) \simeq \partial(V/X) \otimes_{\mathbf{K}} AO_X(\Delta).$$

$\square$

Proof of Theorem 1.3. By Proposition 2.2, $C_X(\Delta)$ is generated over $\partial(V)$ by $AO_X(\Delta)$. Since

$$C(\Delta) = \bigoplus_{X \in L} C_X(\Delta) \quad \text{(Proposition 2.1)},$$

and

$$AO(\Delta) = \bigoplus_{X \in L} AO_X(\Delta),$$

the $\partial(V)$-module $C(\Delta)$ is generated by $AO(\Delta)$. So $\mathcal{B}$ generates $C(\Delta)$ over $\partial(V)$. Define

$$J(\Delta) := \sum_{\varepsilon \in \mathbf{E}^d(\Delta)} \mathbf{K}(\prod \varepsilon)^{-1},$$

which is an ideal of $C(\Delta)$. Then it is known by [OrT2, Theorem 4.2] that

$$C(\Delta) = J(\Delta) \oplus AO(\Delta) \quad \text{(as } \mathbf{K}\text{-vector spaces)}.$$

It is obvious to see that

$$\partial(V)_+ C(\Delta) \subseteq J(\Delta).$$

On the other hand, we have

$$C(\Delta) = \partial(V) AO(\Delta) = \partial(V)_+ AO(\Delta) + AO(\Delta) = \partial(V)_+ C(\Delta) + AO(\Delta).$$

Combining these, we have (2) and (3) at the same time. By (2), we know that $\mathcal{B}$ minimally generates $C(\Delta)$ over $\partial(V)$, which is (1). $\square$

If $M = \bigoplus_{p \geq 0} M_p$ is a graded vector space with $\dim M_p < +\infty \quad (p \geq 0)$, we let

$$\mathrm{Poin}(M, t) = \sum_{p=0}^{\infty} (\dim M_p) t^p$$

be its **Poincaré** (or **Hilbert) series**. Recall [OrT1, 2.42] the (one variable) Möbius function $\mu : L(\Delta) \to \mathbf{Z}$ defined by $\mu(V) = 1$ and for $X > V$ by $\sum_{Y \leq X} \mu(Y) = 0$. Then the **Poincaré polynomial** $\mathrm{Poin}(\mathcal{A}(\Delta), t)$ of the arrangement $\mathcal{A}(\Delta)$ is defined by

$$\mathrm{Poin}(\mathcal{A}(\Delta), t) = \sum_{X \in L} \mu(X)(-t)^{\mathrm{codim}\, X}.$$

Proposition 2.3. ([OrT2, Theorem 4.3]) *For $X \in L$ we have*

$$\dim AO_X(\Delta) = (-1)^{\mathrm{codim}\, X} \mu(X) \ \textit{and} \ \mathrm{Poin}(AO(\Delta), t) = \mathrm{Poin}(\mathcal{A}(\Delta), t).$$

Recall $C(\Delta)$ is a graded $\partial(V)$-module. Since $C(\Delta)$ is infinite dimensional, $\mathrm{Poin}(C(\Delta),t)$ is a formal power series. We now prove Theorem 1.4 which gives a combinatorial formula for $\mathrm{Poin}(C(\Delta),t)$.

Proof of Theorem 1.4. We have

$$\mathrm{Poin}(C(\Delta),t) = \sum_{X\in L} \mathrm{Poin}(C_X(\Delta),t) = \sum_{X\in L} \mathrm{Poin}(\partial(V/X),t)\mathrm{Poin}(AO_X(\Delta),t)$$

by Propositions 2.1 and 2.2. Since the $\mathbf{K}$-algebra $\partial(V/X)$ is isomorphic to the polynomial algebra with $\operatorname{codim} X$ variables, we have

$$\mathrm{Poin}(C(\Delta),t) = \sum_{X\in L} (1-t)^{-\operatorname{codim} X}\mathrm{Poin}(AO_X(\Delta),t).$$

By Proposition 2.3, we have

$$\mathrm{Poin}(AO_X(\Delta),t) = (-1)^{\operatorname{codim} X}\mu(X)t^{\operatorname{codim} X}.$$

Thus

$$\begin{aligned}\mathrm{Poin}(C(\Delta),t) &= \sum_{X\in L} (-1)^{\operatorname{codim} X}\mu(X)\left(\frac{t}{1-t}\right)^{\operatorname{codim} X} \\ &= \mathrm{Poin}(\mathcal{A}(\Delta),(1-t)^{-1}t). \quad \Box\end{aligned}$$

Let Der be the S-module of derivations :

$$\mathrm{Der} = \{\theta \mid \theta : S \to S \text{ is a } \mathbf{K}\text{-linear derivations}\}.$$

Then Der is naturally isomorphic to $S\bigotimes_{\mathbf{K}} V$. Define

$$D(\Delta) = \{\theta \in \mathrm{Der} \mid \theta(\alpha) \in \alpha S \text{ for any } \alpha \in \Delta\},$$

which is naturally an S-submodule of Der. We say that the arrangement $\mathcal{A}(\Delta)$ is **free** if $D(\Delta)$ is a free S-module [OrT1, Definition 4.15]. An element $\theta \in D(\Delta)$ is said to be **homogeneous of degree** p if

$$\theta(x) \in S_p \text{ for all } x \in V^*.$$

When $\mathcal{A}(\Delta)$ is a free arrangement, let $\theta_1, \cdots, \theta_\ell$ be a homogeneous basis for $D(\Delta)$. The ℓ nonnegative integers $\deg\theta_1, \cdots, \deg\theta_\ell$ are called the **exponents** of $\mathcal{A}(\Delta)$. Then one has

Proposition 2.4. (Factorization Theorem [Ter1], [OrT1, Theorem 4.137]) *If $\mathcal{A}(\Delta)$ is a free arrangement with exponents $d_1, \cdots, d_\ell$, then*

$$\mathrm{Poin}(A(\Delta),t) = \prod_{i=1}^{\ell}(1+d_i t).$$

By Theorem 1.4 and Proposition 2.4, we immediately have Corollary 1.5.

The arrangement $\mathcal{A}(\Delta)$ is **generic** if $|\Delta| \geq \ell$ and any ℓ vectors in Δ are linearly independent. In this case, it is easy to see that [OrT1, Lemma 5.122]

$$\mathrm{Poin}(\mathcal{A}(\Delta), t) = (1+t)\sum_{i=0}^{\ell-1}\binom{|\Delta|-1}{i}t^i.$$

Proof of Corollary 1.7. By Theorem 1.4, one has

$$\begin{aligned}
\mathrm{Poin}(C(\Delta), t) &= (1+\frac{t}{1-t})\sum_{i=0}^{\ell-1}\binom{|\Delta|-1}{i}\left(\frac{t}{1-t}\right)^i \\
&= (1-t)^{-\ell}\sum_{i=0}^{\ell-1}(1-t)^{\ell-i-1}\binom{|\Delta|-1}{i}t^i \\
&= (1-t)^{-\ell}\sum_{i=0}^{\ell-1}\binom{|\Delta|-1}{i}t^i\sum_{j=0}^{\ell-i-1}\binom{\ell-i-1}{j}(-1)^j t^j \\
&= (1-t)^{-\ell}\sum_{k=0}^{\ell-1}t^k\sum_{j=0}^{k}(-1)^j\binom{|\Delta|-1}{k-j}\binom{\ell-k+j-1}{j}.
\end{aligned}$$

On the other hand, we have

$$\sum_{j=0}^{k}(-1)^j\binom{|\Delta|-1}{k-j}\binom{\ell-k+j-1}{j} = \binom{|\Delta|-\ell+k-1}{k}$$

by equating the coefficients of x^k in $(1+x)^{|\Delta|-\ell+k-1}$ and $(1+x)^{|\Delta|-1}(1+x)^{-(\ell-k)}$. This proves the assertion. $\square$

We now consider the nbc (=no broken circuit) bases [Bjo1] [Bjo2] [BjZ] [JaT] [OrT2, p.72]. Suppose that Δ is linearly ordered : $\Delta = \{\alpha_1, \cdots, \alpha_n\}$. Let $X \in L$ with $\operatorname{codim} X = p$. Define

$$\begin{aligned}
\mathbf{nbc}_X(\Delta) := \{\varepsilon \in \mathbf{E}_X(\Delta) \mid \varepsilon = (\alpha_{i_1}, \cdots, \alpha_{i_p}), i_1 < \cdots < i_p, \\
\text{contains no broken circuits}\}.
\end{aligned}$$

Let $\mathcal{B}_X = \{(\prod \varepsilon)^{-1} \mid \varepsilon \in \mathbf{nbc}_X(\Delta)\}$ for $X \in L$. Then we have

Proposition 2.5. ([OrT2, Theorem 5.2]) *Let $X \in L$. The set $\mathcal{B}_X$ is a $\mathbf{K}$-basis for $AO_X(\Delta)$.*

Thanks to Propositions 2.1, 2.2 and 2.5 we easily have

Proposition 2.6. *Let $\mathcal{B} = \bigcup_{X\in L}\mathcal{B}_X = \{\phi_1, \cdots, \phi_m\}$. Write $supp(\phi_i) = X$ if $\phi_i \in \mathcal{B}_X$. Then, for any $\phi \in C(\Delta)$ and $j \in \{1, \cdots, m\}$, there uniquely exists $\theta_j \in \partial(V/supp(\phi_j))$ such that*

$$\phi = \sum_{j=1}^{m}\theta_j(\phi_j).$$

Remark 2.7. *Suppose that Δ spans V^* and that $AO_{\{\mathbf{0}\}}(\Delta) = \sum_{j=1}^{q} \mathbf{K}\phi_j$, where $q = |\mu(\{\mathbf{0}\})|$. Then the mapping*

$$\phi \mapsto \sum_{j=1}^{q} \theta_j^{(0)}(\phi_j) \in AO_{\{\mathbf{0}\}}(\Delta)$$

is the restriction to $C(\Delta)$ of the Jeffrey-Kirwan residue [BrV, Definition 6] [Sze]. Here $\theta_j^{(0)}$ is the degree zero part of θ_j $(j = 1, \ldots, q)$.

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